

CfE

ADVANCED Higher MATHEMATICS

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BrightRED Study Guide

CfE

ADVANCED Higher

# MATHEMATICS



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# INTRODUCTION

## INTRODUCING CFE ADVANCED HIGHER MATHEMATICS

### THE STRUCTURE AND AIM OF THIS BOOK

There is no short-cut to passing any course at Advanced Higher level. To obtain a good pass requires consistent, regular revision over the duration of the course. The aim of this book is to bring together, in one volume, with the aid of examples, concise coverage of the course material. The book should be used in conjunction with your course notes and the knowledge gained from attending classes.

In addition to an index at the end of the book, there is a summary skills list which details, in the order of the syllabus given by the SQA, each assessment standard of the course – unit by unit. There is a box alongside, which you might use to monitor and check your progress, perhaps as follows:

an empty box	I have not learned this
• a dot	we have covered this in class
- a dash	I understand this topic, but more practice is required
+ a cross	I have revised this area and I am confident with it.

On the website, you will find further tables (shorter summative versions) which may also be used to track your progress.

Regularly throughout this book are other features:

**Don't Forget** boxes flag up vital pieces of information that you need to remember and important things that you must be able to do, plus some helpful hints.

**Things to Do and Think About** sections contain practice questions to test your understanding. The solutions to these tasks are available on the *Bright Red Digital Zone* ([www.brightredbooks.net](http://www.brightredbooks.net)).

Online references and tests direct you to the *Bright Red Digital Zone*, an online source of examples and solutions covering the entire syllabus. The examples range in length and complexity, and include questions in the same style and format as you will meet in your exams. Solutions will be provided to all examples. This is a useful resource as you study throughout the session.

**Websites and video links** can be used to enhance your knowledge. In addition, [www.hsn.uk.net](http://www.hsn.uk.net) has summary notes, and the SQA website ([www.sqa.org.uk](http://www.sqa.org.uk)) has information on the examination, past papers and solutions which may be useful.

Do remember that success in this hierarchical subject depends on your knowing and using skills from lower levels, such as Higher. Trigonometric identities need to be remembered as well as all the algebraic and calculus techniques you met.

### COURSE STRUCTURE

The Advanced Higher Mathematics course, as outlined in the SQA course specification (<http://www.sqa.org.uk/sqa/48507.html>), is divided into three sections (CAL, APN, MVC):

Calculus	Algebra, Proof and Number Theory	Matrices, Vectors and Complex Numbers
<ul style="list-style-type: none"> <li>Differentiation</li> <li>Integration</li> <li>Differential equations</li> <li>Calculus problems in context</li> </ul>	<ul style="list-style-type: none"> <li>Partial fractions</li> <li>Binomial theorem</li> <li>Sequences and series</li> <li>Summation formulae</li> <li>Functions</li> <li>Direct and indirect proof (contradiction and contrapositive)</li> <li>Proof by induction</li> <li>Tuclid's algorithm</li> </ul>	<ul style="list-style-type: none"> <li>Systems of linear equations</li> <li>Matrices</li> <li>Vectors in three dimensions</li> <li>Complex numbers</li> <li>Geometric operations on complex numbers</li> </ul>

### ASSESSMENT

The Advanced Higher Mathematics course is assessed solely through an external Course Assessment.

The Course Assessment is a written examination consisting of two papers:

Paper 1 (Non-calculator) lasts 1 hour and has 35 marks

Paper 2 (Calculator) lasts 2 hours 30 minutes and has 80 marks

Both papers consist of short and extended response questions.

Paper 2 will usually have more questions of a longer nature than Paper 1.

### EXAMINATION HINTS

You do not need to answer the questions in order. It may be better to choose a question that you can answer easily, so that you settle your nerves. However, remember that the examiners will have attempted to put the shorter questions in order of difficulty, followed by the longer ones, also in order of difficulty. Thus question 12 might well be harder than question 16.

Remember: 100 marks in three hours, that is 1.8 minutes for every mark, or 18 minutes for a long question.

You will not be told in every question to 'show your working', but you need to remember to be accurate, to give detail and to illustrate your understanding in your working. However, you should simplify expressions and try to use concise and efficient methods where possible.

Rigour in mathematics is extremely important, as is communicating your conclusions and results with clarity.

### REVISION TIPS

#### General advice

- **Don't leave your revision until the last minute.** When you are still learning new topics, revise the ones you have already covered.
- Study for periods of between 30 and 45 minutes, unless you are doing a complete paper.
- **Take short breaks,** away from your study area, to keep your level of concentration high.
- During your study leave, build treats and relaxation time into your revision timetable. This will help you to focus and help you stick to your plan.
- In the run-up to the exams, **eat well, exercise well and sleep well.**

#### Maths-specific revision tips

- The best way to revise mathematics is by doing it. There is a time for learning the necessary formulae and rules, but there is no substitute for practice.
- Once you have learned a topic or skill, try questions. Start off with straightforward questions, then Unit level, and progress to examination style. You want to test your knowledge of a topic by trying that discrete area of the syllabus, but you should progress to trying a mixture of past-paper questions. It is important that you start to recognise what to use and when, which skill to apply and where.
- Use the space between **Don't Forgets** in the margin of this book to add your own revision reminders.
- Mathematics is a subject to be practised often. Try to get into the habit of regularly doing mathematics. If you complete one extra question every night in addition to your normal homework and study time at school, you will reap the rewards. You will be able to ask for help the next day when the problem is fresh in your mind, so that you can tackle another question the next evening, and so you will quickly build up your knowledge and confidence.
- Mathematics also demands perseverance. There will come a time when you need to tackle a number of questions or a whole examination paper in one sitting. Time management is essential.
- Mathematics is different from other subjects in so many ways, and the good thing about revising it is that you can be active.

The best way to revise maths is to actively do it.

## ALGEBRA

PARTIAL FRACTIONS **APN**

Certain types of rational functions  $\frac{p(x)}{q(x)}$ , where  $p$  and  $q$  are polynomials in  $x$ , can be decomposed into partial fractions. This can be useful for integrating or differentiating this type of function. For examination purposes,  $q$  can be:

- quadratics or cubics which can easily be factorised into linear factors, or
- cubics which can be factorised into a product of a linear factor and an irreducible quadratic factor. The quadratic/cubic may have a repeated linear factor.

## LINEAR FACTORS

## Example: 1.1

Express  $\frac{x-5}{x^2-x-2}$  in partial fractions.

## Solution:

$$\frac{x-5}{x^2-x-2} = \frac{x-5}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$

Multiplying through by  $(x+1)(x-2)$  gives

$$x-5 = A(x-2) + B(x+1)$$

$x=2$  gives  $B=-1$ ,  $x=-1$  gives  $A=2$ , so:

$$\frac{x-5}{x^2-x-2} = \frac{2}{x+1} - \frac{1}{x-2}$$

## LONG DIVISION

To obtain the partial fraction decomposition of  $\frac{p(x)}{q(x)}$  when the degree of  $p$  is  $\geq$  the degree of  $q$ , use long division to obtain a remainder of the form  $\frac{r(x)}{q(x)}$  where the degree of  $r$  is  $<$  the degree of  $q$ .

Partial fractions can now be performed on  $\frac{r(x)}{q(x)}$ .

## Using long division

## Example: 1.2

Express  $\frac{2x^2-x-9}{x^2-x-2}$  in partial fractions.

## Solution:

$$\frac{2x^2-x-9}{x^2-x-2} = 2 + \frac{x-5}{x^2-x-2} \quad \text{Long division}$$

$$\begin{array}{r} x^2 - x - 2 \overline{) 2x^2 - x - 9} \\ \underline{2x^2 - 2x - 4} \phantom{0} \\ 2x - 5 \phantom{0} \end{array}$$

Because the highest power of  $x$  is now less than 2, the process stops, giving the result shown. Alternatively, this could be done by inspection.

Now proceeding as in example 1.1, we get:

$$\frac{2x^2-x-9}{x^2-x-2} = 2 + \frac{2}{x+1} - \frac{1}{x-2}$$

**DON'T FORGET**

Long division is required for  $\frac{p(x)}{q(x)}$  when  $\deg p \geq \deg q$ .

## REPEATED FACTOR

## Example: 1.3

Express  $\frac{3x^2-11x+4}{(x+1)(x-2)^2}$  in partial fractions.

## Solution:

$$\frac{3x^2-11x+4}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$3x^2 - 11x + 4 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$$

$x=-1$  gives  $A=2$ ,  $x=2$  gives  $C=-2$ .

Comparing the coefficient of  $x^2$  on both sides gives  $3 = A + B$ , from which  $B = 1$ . Therefore,

$$\frac{3x^2-11x+4}{(x+1)(x-2)^2} = \frac{2}{x+1} + \frac{1}{x-2} - \frac{2}{(x-2)^2}$$

## IRREDUCIBLE FACTOR

If  $q(x)$  contains an irreducible factor such as  $x^2 + a^2$ , you need  $\frac{Ax+B}{x^2+a^2}$ .

For  $x^2 + x + 1$ , you need  $\frac{Ax+B}{x^2+x+1}$ .

## Example: 1.4

Express  $\frac{2x^2+x+27}{x^3+9x}$  in partial fractions.

## Solution:

$$\frac{2x^2+x+27}{x^3+9x} = \frac{A}{x} + \frac{Bx+C}{x^2+9}$$

$$\therefore 2x^2 + x + 27 = A(x^2 + 9) + x(Bx + C)$$

Comparing coefficients:

$$x^2: 2 = A + B$$

$$x: 1 = C$$

$$x^0: 27 = 9A$$

$$\therefore A = 3, B = -1, C = 1$$

$$\frac{2x^2+x+27}{x^3+9x} = \frac{3}{x} + \frac{-x+1}{x^2+9} \quad \left[ \text{or } \frac{3}{x} + \frac{1-x}{x^2+9} \right]$$

**THINGS TO DO AND THINK ABOUT**

- Express  $\frac{x-2}{3x^2-10x+3}$  in partial fractions.
- Express  $\frac{1}{x^2-4x}$  in partial fractions.
- Express  $\frac{x^2-x^2+2}{(x+1)^2}$  in partial fractions.
- Express  $\frac{3x^2-13x+50}{(x-2)(x-3)}$  in the form  $A + \frac{B}{x-2} + \frac{C}{x-3}$  where  $A, B, C \in \mathbb{Z}$ . State the values of  $A, B$  and  $C$ .
- Describe the behaviour of the graph of function  $f(x) = \frac{2x^2+3x^2-14x+k}{x^2+4x+3}$  where  $k \in \mathbb{N}$  for large, positive values of  $x$ .
- Express  $\frac{x^2+2}{x(x^2-1)}$  in partial fractions.

**DON'T FORGET**

If  $q(x)$  contains a factor  $(x-a)^2$ , you need  $\frac{A}{x-a} + \frac{B}{(x-a)^2}$

**DON'T FORGET**

Use  $Ax+B$  for irreducible quadratics.

**ONLINE**

Learn more about partial fractions by following the link at [www.brightonbooks.net](http://www.brightonbooks.net)

**VIDEO LINK**

Check out the clip at [www.brightonbooks.net](http://www.brightonbooks.net) for more on this topic.

**ONLINE TEST**

Test yourself on partial fractions at [www.brightonbooks.net](http://www.brightonbooks.net)

# MATHEMATICS

Linda Moon, Philip Moon and Dr Michael Green

This BrightRED Study Guide is the ultimate companion to your CFE Advanced Higher Mathematics studies! Written by our trusted authors and experienced mathematics teachers, Linda Moon, Philip Moon and Dr Michael Green, this book is full-colour and packed with clear and accessible information, excellent examples, activities and advice. Inside, you will find:

- ▶ All the essential course information, fully up-to-date with SQA course changes, arranged in easily digestible double-page topic spreads.
- ▶ Detailed full-colour diagrams, illustrations and data boxes to make sure all that study sticks!
- ▶ **Don't forget** pointers offering advice on the key facts to remember, and on how to avoid common mistakes.
- ▶ **Things to do and think about** sections encouraging the regular review of key points covered.
- ▶ **Digital Zone** activities and tests to supercharge your learning efforts online!
- ▶ An **index of key terms** to help when revising.

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