# ADVANCED HIGHER MATHEMATICS REVISION: *NOW TRY THIS* ANSWER SECTION

## CHAPTER 1

### Factorials and binomial coefficients – page 7

(1) $$\binom{n+1}{3}-\binom{n}{3}=\frac{(n+1)!}{3!(n-2)!}-\frac{n!}{3!(n-3)!}$$
$$=\frac{(n+1)!}{3!(n-2)!}-\frac{n!(n-2)}{3!(n-2)!}$$
$$=\frac{(n+1)!-n!(n-2)}{3!(n-2)!}$$
$$=\frac{n![(n+1)-(n-2)]}{3!(n-2)!}$$
$$=\frac{n!\times 3}{3!(n-2)!}=\frac{n!}{2!(n-2)!}$$
$$=\binom{n}{2}$$

(2) $$2\times\frac{n(n-1)(n-2)(n-3)}{24}=\frac{n(n-1)}{2}$$

Dividing by $n(n-1)$ (NB $n\geq 4$) gives $(n-2)(n-3)=6$.

Expanding gives $n = 5$.

(3) $$\binom{n}{n-3}+\binom{n+1}{3}=\frac{n!}{(n-3)!3!}+\frac{(n+1)!}{(n+1-3)!3!}$$

Also, $(n + 1)! = (n + 1)n!$ and $(n - 2)! = (n - 2)(n - 3)!$ So RHS is

$$\frac{n!}{(n-3)!3!}+\frac{(n+1)n!}{(n-2)(n-3)!3!}=\frac{n!}{(n-3)!3!}\left(1+\frac{n+1}{n-2}\right)$$

$$=\frac{n!}{(n-3)!3!}\times\frac{2n-1}{n-2}=\frac{n!(2n-1)}{(n-2)!3!}=\frac{n(n-1)(2n-1)}{6}.$$

### The binomial theorem – page 9

(1) $$(a^3+2)^4=(a^3)^4+\binom{4}{1}(a^3)^3 2+\binom{4}{2}(a^3)^2 2^2+\binom{4}{3}(a^3)2^3+2^4$$
$$=a^{12}+8a^9+24a^6+32a^3+16.$$

(2) $r^{\text{th}}$ term is $$\binom{10}{r}(x^2)^{10-r}\left(\frac{1}{x}\right)^r=\binom{10}{r}x^{20-3r}$$

Requested term is $45x^{14}$

(3) $$\left(z-\frac{1}{z}\right)^5=z^5+5z^4\left(-\frac{1}{z}\right)+10z^3\left(-\frac{1}{z}\right)^2+10z^2\left(-\frac{1}{z}\right)^3$$
$$+5z\left(-\frac{1}{z}\right)^4+\left(-\frac{1}{z}\right)^5$$
$$=z^5-5z^3+10z-\frac{10}{z}+\frac{5}{z^3}-\frac{1}{z^5}$$
$$=\left(z^5-\frac{1}{z^5}\right)-5\left(z^3-\frac{1}{z^3}\right)+10\left(z-\frac{1}{z}\right)$$

$\therefore A=1,\quad B=-5,\quad C=10.$

(4) General term is $$\binom{6}{r}(3x)^r\left(-\frac{2}{x^2}\right)^{6-r}=\binom{6}{r}3^r(-2)^{6-r}x^{3r-12}.$$

Need $x^0$ so $3r - 12 = 0$ and $r = 4$. Constant term

$$\binom{6}{4}3^4(-2)^2=\frac{6\times 5}{2}\times 81\times 4=4860.$$

(5) Coefficient of $x^r$ in $(1+x)^{n+1}$ is $\binom{n+1}{r}$.

RHS is $(1+\cdots+\binom{n}{r-1}x^{r-1}+\binom{n}{r}x^r+\cdots)(1+x)$ so coefficient

of $x^r$ is $\binom{n}{r}+\binom{n}{r-1}$.

Only use this method when asked to.

(6) From $\binom{n+1}{r}=\binom{n}{r}+\binom{n}{r-1}$ we have $\binom{n+1}{r+1}=\binom{n}{r}+\binom{n}{r+1}$

$(r\to r+1)$,

and $\binom{n+1}{r+2}=\binom{n}{r+1}+\binom{n}{r+2}$ $(r\to r+2)$. Adding these two

results gives

$$\binom{n}{r}+2\binom{n}{r+1}+\binom{n}{r+2}=\binom{n+1}{r+1}+\binom{n+1}{r+2}=\binom{n+2}{r+2}.$$

(7) Set $x = 1$ in $(1+x)^n=1+\binom{n}{1}x+\cdots+\binom{n}{n-1}x^{n-1}+x^n$.

### Partial fractions – page 11

(1) $$\frac{x-2}{3x^2+10x+3}=\frac{x-2}{(3x+1)(x+3)}=\frac{A}{3x+1}+\frac{B}{x+3}$$

$x-2=A(x+3)+B(3x+1);$

$x=-3\Rightarrow -5=-8B,\quad B=\frac{5}{8}.$

$x=-\frac{1}{3}\Rightarrow -2\frac{1}{3}=2\frac{2}{3}A,\quad A=-\frac{7}{8}$ or

Equating coefficients of $x$: $A+3B=1\Rightarrow A=1-\frac{15}{8}=-\frac{7}{8}$

$$\therefore\frac{x-2}{3x^2+10x+3}=\frac{5}{8(x+3)}-\frac{7}{8(3x+1)}$$

(2) $$\frac{1}{x^3+4x}=\frac{1}{x(x^2+4)}=\frac{A}{x}+\frac{Bx+C}{x^2+4}$$

$1=A(x^2+4)+Bx^2+Cx;\ x=0\Rightarrow 1=4A,\quad A=\frac{1}{4}$

equating coefficients of $x^2$ and $x$: $0=A+B,\quad 0=C$.

$A=\frac{1}{4},\quad B=-\frac{1}{4},\quad C=0.$

$$\therefore\frac{1}{x^3+4x}=\frac{1}{4x}-\frac{x}{4(x^2+4)}$$

(3) Dividing out, $\frac{x^3+x^2+2}{(x+1)^2} = x-1+\frac{x+3}{(x+1)^2}$.

Let $\frac{x+3}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$.

$x+3 = A(x+1)+B$; $x=-1$ gives $B=2$; coefficient of $x$ gives $A=1$.

Hence, $\frac{x^3+x^2+2}{(x+1)^2} = x-1+\frac{1}{x+1}+\frac{2}{(x+1)^2}$.

## CHAPTER 2

### Standard differentials – page 13

(1) $f'(x) = 2\sin x\cos x\, e^{-\tan x} + \sin^2 x(-\sec^2 x)e^{-\tan x}$

$= (\sin 2x - \tan^2 x)\, e^{-\tan x}$;

$f'(\frac{\pi}{4}) = (\sin\frac{\pi}{2} - \tan^2(\frac{\pi}{4}))\, e^{-\tan(\frac{\pi}{4})} = (1-1)e^{-1} = 0$.

(2) $f'(x) = 2x\tan 3x + x^2 3\sec^2 3x$

$= x(2\tan 3x + 3x\sec^2 3x)$

$= x\sec 3x(2\sin 3x + 3x\sec 3x)$.

(3) $\frac{dy}{dx} = \frac{(1+2x)e^x - 2e^x}{(1+2x)^2}$

$= \frac{(2x-1)e^x}{(1+2x)^2}$,

$\frac{dy}{dx} = 0$ when $(2x-1)e^x = 0$, $e^x \neq 0$, $\therefore x = \frac{1}{2}$.

(4) $x = T^3 - 90T^2 + 2400T$

$\frac{dx}{dT} = 3T^2 - 180T + 2400$

$= 0$ at stationary values.

$3(T-40)(T-20) = 0$

i.e. $T = 20$ or $T = 40$

$\frac{d^2x}{dT^2} = 6T - 180$

$\Rightarrow T = 20$ is a local maximum (and $T = 40$ is a local minimum). $x(20) = 20000$

Check end of interval values

$x(10) = 1000 - 9000 + 24000 = 16000$

$x(60) = 216000 - 324000 + 144000 = 36000$

So the best result is when $T = 60$.

### Further differentiation 1 – page 15

(1) (a) $f(x) = (2+x)\tan^{-1}\sqrt{x-1}$

$$f'(x) = \tan^{-1}\sqrt{x-1} + (2+x)\frac{1}{1+(\sqrt{x-1})^2}\left(\frac{1}{2}\cdot\frac{1}{\sqrt{x-1}}\right)$$

$$= \tan^{-1}\sqrt{x-1} + \frac{2+x}{2x\sqrt{x-1}}.$$

(b) $g(x) = e^{\cot 2x}$

$g'(x) = -2\mathrm{cosec}^2 2x e^{\cot 2x}$.

(2) Differentiating wrt $x$: $2y\frac{dy}{dx} + 3y + 3x\frac{dy}{dx} = 2x$

$\frac{dy}{dx} = \frac{2x-3y}{2y+3x}$

At P(–1, 1): $\frac{dy}{dx} = \frac{-2-3}{2-3} = 5$, gradient of the tangent is 5.

or

If the gradient of the tangent to the curve at $P$ is $m$,

$2(1)m + 3(1) + 3(-1)(1)m = -2$, i.e. $-m = -5 \Rightarrow m = 5$.

(3) $xy - x = 4$

$\frac{d}{dx}(xy) - 1 = 0$

$x\frac{dy}{dx} + y - 1 = 0$

$\frac{dy}{dx} = \frac{1-y}{x}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1-y}{x}\right)$

$= \frac{-x\frac{dy}{dx} - (1-y)}{x^2}$

$= \frac{-x\left(\frac{1-y}{x}\right) - (1-y)}{x^2}$

$= \frac{2(y-1)}{x^2}$

### Further differentiation 2 – page 17

(1) $y = (x+1)^2(x+2)^{-4}$

$\ln y = 2\ln(x+1) - 4\ln(x+2)$

$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x+1} - \frac{4}{x+2}$

$\frac{dy}{dx} = \left(\frac{2}{x+1} - \frac{4}{x+2}\right)y$

$a = 2$; $b = -4$.

(2) (a) $f'(x) = 3\sin 3x\exp(-\cos 3x)$,

(b) $\ln y = \ln 2^{(x^3+x)}$

$\ln y = (x^3+x)\ln 2$

$\frac{1}{y}\frac{dy}{dx} = (3x^2+1)\ln 2$

so $\frac{dy}{dx} = y(3x^2+1)\ln 2$

$= \ln 2(3x^2+1)2^{x^3+x}$.

(3) $f'(x) = \frac{1}{\sqrt{1-(4x)^2}}.4$

$= \frac{4}{\sqrt{1-16x^2}}$

(4) $\frac{dx}{d\theta}=\sec^2\theta,\quad \frac{dy}{d\theta}=\cos\theta$

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\cos\theta}{\sec^2\theta}=\cos^3\theta$$

(5) When depth is $x$, radius = $x\tan\frac{\pi}{3}=x\sqrt{3}$

so $V=\frac{\pi}{3}(x\sqrt{3})^2x=\pi x^3$.

Hence $\frac{dV}{dt}=3\pi x^2\frac{dx}{dt}$ and we are given $\frac{dV}{dt}=2$

so $3\pi x^2\frac{dx}{dt}=2$.

When $x$ = 3 $3\pi 3^2\frac{dx}{dt}=2\Rightarrow\frac{dx}{dt}=\frac{2}{27\pi}$, so depth increasing at $\frac{2}{27\pi}$ cm per second.

## CHAPTER 3

### Basic integration – page 19

(1) $x=(u-1)^2\Rightarrow dx=2(u-1)du$

$$\int\frac{1}{(1+\sqrt{x})^3}dx=\int\frac{2(u-1)}{u^3}du$$

$$=2\int(u^{-2}-u^{-3})du$$

$$=2\left(\frac{-1}{u}+\frac{1}{2u^2}\right)+c$$

$$=\frac{1}{(1+\sqrt{x})^2}-\frac{2}{1+\sqrt{x}}+c$$

### Further integration – page 20

(1) $1+x^2=u\Rightarrow 2x\,dx=u$

$x=0\Rightarrow u=1;\ x=1\Rightarrow u=2$

$$\int_0^1\frac{x^3}{(1+x^2)^4}dx$$

$$=\int_1^2\frac{(u-1)}{2u^4}du$$

$$=\frac{1}{2}\int_1^2(u^{-3}-u^{-4})du$$

$$=\frac{1}{2}\left[-\frac{1}{2}u^{-2}+\frac{1}{3}u^{-3}\right]_1^2$$

$$=\frac{1}{24}$$

Hence the volume of revolution = $\frac{\pi}{24}$

(2) $x=2\sin\theta\Rightarrow dx=2\cos\theta\,d\theta$

$x=0\Rightarrow\theta=0;\ x=\sqrt{2}\Rightarrow\sin\theta=\frac{1}{\sqrt{2}}\Rightarrow\theta=\frac{\pi}{4}$

$$\int_0^{\sqrt{2}}\frac{x^2}{\sqrt{4-x^2}}dx=\int_0^{\pi/4}\frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}(2\cos\theta)d\theta$$

$$=\int_0^{\pi/4}\frac{4\sin^2\theta}{2\cos\theta}(2\cos\theta)d\theta$$

$$=2\int_0^{\pi/4}(2\sin^2\theta)d\theta$$

$$=2\int_0^{\pi/4}(1-\cos2\theta)d\theta$$

$$=2\left[\theta-\frac{1}{2}\sin2\theta\right]_0^{\pi/4}$$

$$=2\left\{\left[\frac{\pi}{4}-\frac{1}{2}\right]-0\right\}$$

$$=\frac{\pi}{2}-1$$

### Page 21

(1) (a) $\frac{1}{x^2+2x-8}=\frac{1}{(x-2)(x+4)}=\frac{A}{x-2}+\frac{B}{x+4}$

$1=A(x+4)+B(x-2);\quad x=2\Rightarrow A=\frac{1}{6};\quad x=-4\Rightarrow B=-\frac{1}{6}$.

(b) $\int_0^1\frac{1}{x^2+2x-8}dx=\frac{1}{6}\int_0^1\left(\frac{1}{x-2}-\frac{1}{x+4}\right)dx$

$$=\frac{1}{6}\left[\ln|x-2|-\ln|x+4|\right]_0^1$$

$$=\frac{1}{6}\left[\ln1-\ln5-\ln2+\ln4\right]$$

$$=\frac{1}{6}(\ln2-\ln5)$$

$$=\frac{1}{6}\ln\frac{2}{5}$$

(2) (a) $\frac{1}{x^3+x}=\frac{1}{x}-\frac{x}{x^2+1}$ (see solutions for chapter 1)

(b) $I(k)=\int_1^k\frac{1}{x^3+x}dx=\int_1^k\left(\frac{1}{x}-\frac{x}{x^2+1}\right)dx$

$$=\int_1^k\frac{1}{x}dx-\int_1^k\frac{x}{x^2+1}dx$$

$$=[\ln x]_1^k-\frac{1}{2}\left[\ln(x^2+1)\right]_1^k$$

$$=[\ln k-0]-\frac{1}{2}\left[\ln(k^2+1)-\ln2\right]$$

$$=\ln k-\ln\sqrt{k^2+1}+\ln\sqrt{2}$$

$$=\ln\frac{k\sqrt{2}}{\sqrt{k^2+1}}.$$

### Integration by parts – page 23

(1) $\int10x^2\cos5x\,dx=10x^2\frac{\sin5x}{5}-\int20x\frac{\sin5x}{5}dx$

$$=2x^2\sin5x-4\int x\sin5x\,dx$$

$$\int x\sin5x\,dx=x\frac{\cos5x}{-5}+\int\frac{\cos5x}{5}dx=-\tfrac{1}{5}x\cos5x+\tfrac{1}{25}\sin5x$$

So $\int10x^2\cos5x\,dx=2x^2\sin5x+\frac{4}{5}x\cos5x-\frac{4}{25}\sin5x+c$

(2) $\int_0^1 x\tan^{-1}x^2 dx = \left[\tan^{-1}x^2 \int x\,dx\right]_0^1 - \int_0^1 \frac{2x}{1+x^4}\frac{x^2}{2}dx$

$= \left[\frac{1}{2}x^2\tan^{-1}x^2\right]_0^1 - \int_0^1 \frac{x^3}{1+x^4}dx$

$= \left[\frac{1}{2}x^2\tan^{-1}x^2\right]_0^1 - \left[\frac{1}{4}\ln(1+x^4)\right]_0^1$

$= \frac{1}{2}\tan^{-1}1 - 0 - \left[\frac{1}{4}\ln 2 - \frac{1}{4}\ln 1\right]$

$= \frac{\pi}{8} - \frac{1}{4}\ln 2$

(3) $\int \frac{(x+3)^2}{(x+1)^3}dx = -\frac{(x+3)^2}{2(x+1)^2} + \int \frac{x+3}{(x+1)^2}dx$;

$\int \frac{x+3}{(x+1)^2}dx = -\frac{x+3}{x+1} + \int \frac{1}{x+1}dx$

so $\int \frac{(x+3)^2}{(x+1)^3}dx = -\frac{(x+3)^2}{2(x+1)^2} - \frac{x+3}{x+1} + \ln|x+1| + c$.

(4) $I = \int_0^{\pi/2} e^{2x}\cos x\,dx = \left[e^{2x}\sin x\right]_0^{\pi/2} - 2J$ and

$J = \int_0^{\pi/2} e^{2x}\sin x\,dx = \left[-e^{2x}\cos x\right]_0^{\pi/2} + 2I$

so $I = e^{\pi} - 2J$ and $J = 1 + 2I$. Solving these equations for $I$

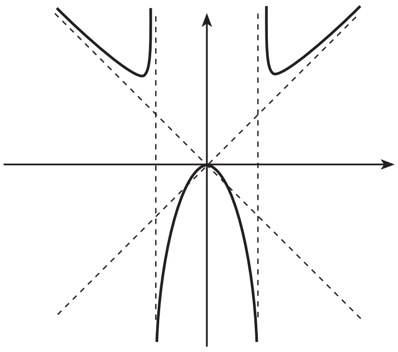
gives $I = \frac{1}{5}(e^{\pi} - 2)$.

## CHAPTER 4

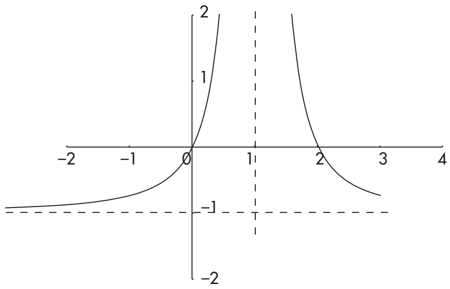
**Properties of functions – page 25**

(1) $f(-x) = (-x)^3\tan(-x) = -x^3(-\tan x) = x^3\tan x$

So $f(x) = f(-x)$ for all $x$, and so $f$ is even.

(2) 

The other asymptotes are $y = -x$ and $x = -1$

(3) 


Asymptotes are $y = -1$ and $x = 1$.

(4) The graph is not symmetrical about the y-axis (or $f(x) \neq f(-x)$) so it is not an even function.
The graph does not have half-turn rotational symmetry (or $f(x) \neq -f(-x)$) so it is not an odd function.
The function is neither even nor odd.

**Critical points – page 29**

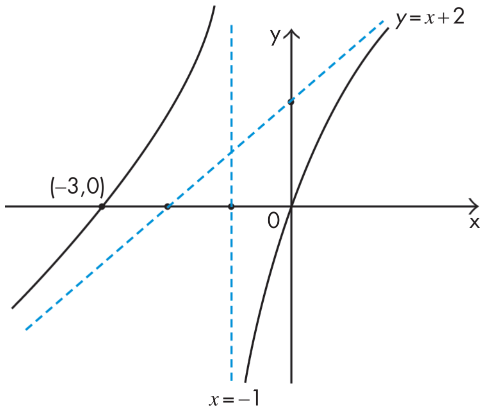
(1) (a) $f(x) = \frac{x^2+3x}{x+1} = x+2-\frac{2}{x+1}$ using long division,

giving oblique asymptote: $y = x + 2$

and vertical asymptote: $x = -1$.

(b) $f'(x) = 1 + \frac{2}{(x+1)^2} > 0$ for all $x \neq -1$, so the graph of $f$ is always increasing.

(c) Sketch of $f$ given in diagram:




(2) (a) $y = \frac{x-3}{x+2} = 1 - \frac{5}{x+2}$

Vertical asymptote is $x = -2$.

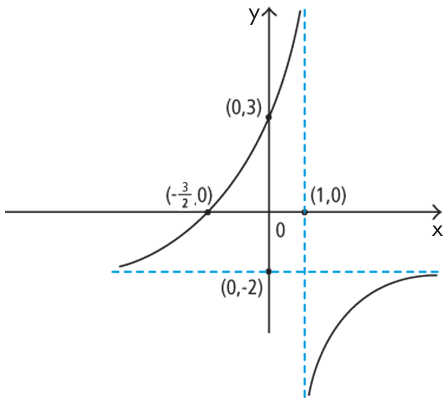
Horizontal asymptote is $y = 1$.

(b) $\frac{dy}{dx} = \frac{5}{(x+2)^2} \neq 0$

so no stationary values.

(c) $\frac{d^2y}{dx^2} = \frac{-10}{(x+2)^3} \neq 0$

so no points of inflexion.

(d) 


The asymptotes are $x = 1$ and $y = -2$.

The domain must exclude $x = 1$.

NB. You are not required to obtain a formula for $f^{-1}$.

# NTT Answers

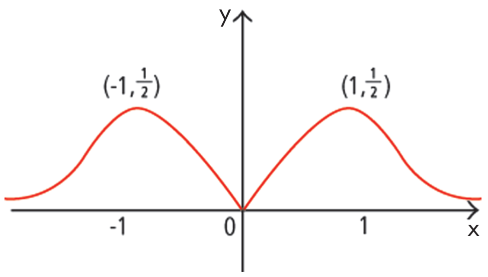
**Related graphs – page 30**

(1) $f(x)=\dfrac{x}{1+x^2}$

$$f'(x)=\frac{(1+x^2)-2x^2}{(1+x^2)^2}$$

$$=\frac{1-x^2}{(1+x^2)^2}$$

The graph of $f(x)$ has stationary values at $(1,\frac{1}{2})$ and $(-1,-\frac{1}{2})$.




Thus $g$ has turning points $(1,\frac{1}{2})$ and $(-1,\frac{1}{2})$ and a critical point (0, 0) (where its gradient is discontinuous).

**Graphs of trigonometric functions – page 31**

(1) (a) $\tan^{-1}2x$ has horizontal asymptotes at $y=\pm\dfrac{\pi}{2}$.

(b) Area

$$=\int_0^{1/2}\tan^{-1}2x\,dx$$

$$=\int_0^{1/2}(\tan^{-1}2x)\times 1\,dx$$

$$=\left[\tan^{-1}2x\int 1.\,dx-\int\frac{2}{1+4x^2}.\,x\,dx\right]_0^{1/2}$$

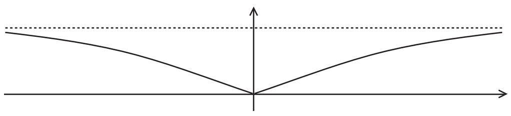
$$=\left[x\tan^{-1}2x-\frac{1}{4}\int\frac{8x}{1+4x^2}dx\right]_0^{1/2}$$

$$=\left[x\tan^{-1}2x-\frac{1}{4}\ln(1+4x^2)\right]_0^{1/2}$$

$$=\left[\frac{1}{2}\tan^{-1}1-\frac{1}{4}\ln 2\right]-[0-0]$$

$$=\frac{\pi}{8}-\frac{1}{4}\ln 2$$

(c)



$$\int_{-1/2}^{1/2}|f(x)|\,dx=2\int_0^{1/2}\tan^{-1}2x\,dx$$

$$=\frac{\pi}{4}-\frac{1}{2}\ln 2$$

## CHAPTER 5

**Systems of linear equations and Gaussian elimination – page 33**

(1) $\left(\begin{array}{ccc|c}2&-1&2&1\\1&1&-2&2\\1&-2&4&-1\end{array}\right)\Rightarrow\left(\begin{array}{ccc|c}2&-1&2&1\\0&-3&6&-3\\0&3&-6&3\end{array}\right)\Rightarrow\left(\begin{array}{ccc|c}2&-1&2&1\\0&-3&-6&-3\\0&0&0&0\end{array}\right)$

Thus $z=t$, $y=1+2t$ and $x=1$.

(2)

$$x+y-z=6$$
$$2x-3y+2z=2$$
$$-5x+2y-4z=1$$

$\left(\begin{array}{ccc|c}1&1&-1&6\\2&-3&2&2\\-5&2&-4&1\end{array}\right)\Rightarrow\left(\begin{array}{ccc|c}0&1&-1&6\\0&-5&4&-10\\0&7&-9&31\end{array}\right)\Rightarrow\left(\begin{array}{ccc|c}0&1&-1&6\\0&-5&4&-10\\0&0&-\frac{17}{5}&17\end{array}\right)$

$$z=17\div\left(\frac{-17}{5}\right)=-5$$

$-5y-20=-10\Rightarrow y=-2$

$x-2+5=6\Rightarrow x=3$

**Special cases – page 35**

(1) (a) $\left(\begin{array}{ccc|c}1&1&3&1\\3&a&1&0\\1&1&1&-1\end{array}\right)\Rightarrow\left(\begin{array}{ccc|c}1&1&3&1\\0&a-3&-8&-3\\0&0&-2&-2\end{array}\right)$

(b) When $a\neq 3$, we can solve to give the unique solution

$$z=1;\quad y=\frac{5}{a-3};\quad x=\frac{1-2a}{a-3}.$$

When $a=3$, the third equation gives $z=1$, but the second equation gives $z=\frac{3}{8}$.

(c) Hence the equations are inconsistent when $a=3$.

(2) (a) $\left(\begin{array}{ccc|c}1&1&2&1\\2&\lambda&1&0\\3&3&9&5\end{array}\right)\Rightarrow\left(\begin{array}{ccc|c}1&1&2&1\\0&\lambda-2&-3&-2\\0&0&3&2\end{array}\right)$

$$z=\tfrac{2}{3};\quad(\lambda-2)y-2=-2\quad\Rightarrow\quad y=0;\quad x=1-0-\tfrac{4}{3}=-\tfrac{1}{3}.$$

(b) When $\lambda=2$, the second and third rows in the reduced matrix are the same, one row is redundant, so there is an infinite number of solutions.

(3) Solving gives $x=2$, $y=0.4$.
Changing two of the coefficients in the original equation by no more than 5% produces a change in $x$ of 200% and a change in y of 87%.
The equations are ill-conditioned.

## CHAPTER 6

**Basic matrix operations – page 39**

(1) (a) $AB=\begin{pmatrix}8&15\\12&3\end{pmatrix}$

(b) $4C+D=\begin{pmatrix}4x+2&15\\12&4y-1\end{pmatrix}$

(c) $4x+2=8$, so $x=3/2$; $4y-1=3$ so $y=1$.

(2) $AB=\begin{pmatrix}1&1&1\\1&2&3\\1&-1&-1\end{pmatrix}\begin{pmatrix}1&0&1\\4&-2&-2\\-3&2&1\end{pmatrix}=\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}$ so $k=2$.

$$A^2B=A(AB)=\begin{pmatrix}1&1&1\\1&2&3\\1&-1&-1\end{pmatrix}\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}=\begin{pmatrix}2&2&2\\2&4&6\\2&-2&-2\end{pmatrix}.$$

(3) (a) $\det\begin{pmatrix}t+4&3t\\3&5\end{pmatrix}=5(t+4)-9t=20-4t$

$$A^{-1}=\frac{1}{20-4t}\begin{pmatrix}5&-3t\\-3&t+4\end{pmatrix}$$

(b) $20-4t=0\Rightarrow t=5$

(c) $\begin{pmatrix}t+4&3\\3t&5\end{pmatrix}=\begin{pmatrix}6&3\\6&5\end{pmatrix}\Rightarrow t=2$

**Special matrices – page 43**

(1) (*a*) $x=\pm 2$

(*b*) When $x=2$, $A=\begin{pmatrix}1&2\\2&4\end{pmatrix}$

$$A^2=\begin{pmatrix}1&2\\2&4\end{pmatrix}\begin{pmatrix}1&2\\2&4\end{pmatrix}=\begin{pmatrix}5&10\\10&20\end{pmatrix}=5A$$

$A^4=(A^2)^2=(5A)^2=25A^2=125A$

(2) $\det\begin{pmatrix}1&2\\-x&3\end{pmatrix}=3+2x$, so the inverse is $\frac{1}{3+2x}\begin{pmatrix}3&-2\\x&1\end{pmatrix}$.

Matrix singular when $3+2x=0\Rightarrow x=-\frac{3}{2}$.

(3) (*a*) $AB=\begin{pmatrix}x&x&x\\-6&6&-1\\0&0&8\end{pmatrix}$

(*b*) $\det A=3$

$\det AB=96x$

$\det B=\dfrac{\det AB}{\det A}$

$\det B=32x$

(4) (a) $\det A=4\lambda-2(\lambda+3)=2\lambda-6$,

so $A^{-1}=\dfrac{1}{2\lambda-6}\begin{pmatrix}4&-2\\-\lambda-3&\lambda\end{pmatrix}$

(b) $2\lambda-6=0\Rightarrow\lambda=3$

(c) $A'=\begin{pmatrix}\lambda&\lambda+3\\2&4\end{pmatrix}=\begin{pmatrix}-2&1\\2&4\end{pmatrix}\Rightarrow\lambda=-2.$

(5) The determinant of the matrix must be zero:

$$\begin{vmatrix}1&1&0\\0&k-2&-1\\1&2&k\end{vmatrix}=1\cdot\begin{vmatrix}k-2&-1\\2&k\end{vmatrix}-1\cdot\begin{vmatrix}0&-1\\1&k\end{vmatrix}+0$$

$$=(k-2)k+2-(0+1)$$

$$=k^2-2k+1=(k-1)^2.$$

Hence the matrix does not have an inverse when $k=1$.

(6) $A^2=\begin{pmatrix}0&4&2\\1&0&1\\-1&-2&-3\end{pmatrix}\begin{pmatrix}0&4&2\\1&0&1\\-1&-2&-3\end{pmatrix}=\begin{pmatrix}2&-4&-2\\-1&2&-1\\1&2&5\end{pmatrix}$

$$A^2+A=\begin{pmatrix}2&-4&-2\\-1&2&-1\\1&2&5\end{pmatrix}+\begin{pmatrix}0&4&2\\1&0&1\\-1&-2&-3\end{pmatrix}=\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}=2I,$$

so $k=2$.

$A^{-1}(A^2+A)=A^{-1}(2I)=2A^{-1}$

$A+I=2A^{-1}$

$A^{-1}=\frac{1}{2}A+\frac{1}{2}I$, so $p=q=\frac{1}{2}$.

**Geometric transformations of the plane – page 45**

(1) $M_1=\begin{pmatrix}\cos\frac{\pi}{2}&-\sin\frac{\pi}{2}\\\sin\frac{\pi}{2}&\cos\frac{\pi}{2}\end{pmatrix}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ and $M_2=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$$M_2M_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$$

The transformation represented by $M_2M_1$ is a reflection in the line $y=-x$.

(2) The matrix $\begin{pmatrix}2&0\\0&2\end{pmatrix}$ gives an enlargement, scale factor 2.

The matrix $\begin{pmatrix}\frac{1}{2}&\frac{\sqrt{3}}{2}\\-\frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix}$ gives a clockwise rotation of 60° about the origin.

$$M=\begin{pmatrix}\frac{1}{2}&\frac{\sqrt{3}}{2}\\-\frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix}\begin{pmatrix}2&0\\0&2\end{pmatrix}$$

$$=\begin{pmatrix}1&\sqrt{3}\\-\sqrt{3}&1\end{pmatrix}.$$

## CHAPTER 7

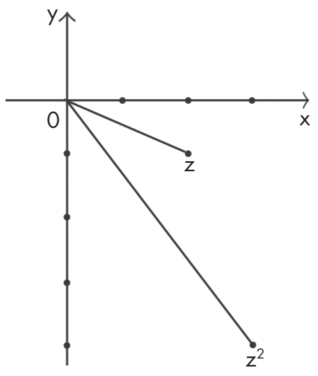
**Introducing complex numbers – page 47**

(1) $\dfrac{1}{z}=\dfrac{1}{2-i}\cdot\dfrac{2+i}{2+i}$

$=\dfrac{2+i}{5}=\dfrac{2}{5}+\dfrac{1}{5}i$

$z^2=4-4i-1=3-4i.$

Argand diagram.




(2) Let $z = x + iy$ so equation is $x + iy + x^2 + y^2 = 7 - i$.

Equating real parts: $x + x^2 + y^2 = 7$.

Equating imaginary parts: $y = -1$

Substituting for $y$ into first equation gives

$$x + x^2 + 1 = 7$$
$$x^2 + x - 6 = 0$$
$$(x+3)(x-2) = 0$$

which gives $x = -3$, $x = 2$.

The solutions are $2 - i$ and $-3 - i$.

(3) Either $z^3 = zz^2 = (2+ai)(2+ai)^2 = (2+ai)(4-4a^2+4ai)$

$$= 8 - 6a^2 + (12a - a^3)i$$

or

$$z^3 = (2+ai)^3 = 2^3 + 3(2^2)ai + 3(2)(ai)^2 + (ai)^3$$
$$= 8 + 12ai - 6a^2 - a^3 i = 8 - 6a^2 + (12a - a^3)i$$

which is real when $12a - a^3 = 0$. As $a > 0$, solving gives $a = \sqrt{12} = 2\sqrt{3}$.

$$|z| = \left|2 + i\sqrt{12}\right| = \sqrt{4+12} = 4,$$

$$\arg(2 + i\sqrt{12}) = \tan^{-1}\left(\frac{\sqrt{12}}{2}\right) = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

**Polar form and de Moivre's theorem – page 49**

(1) (a) $z^k = (\cos\theta + i\sin\theta)^k$
$= \cos k\theta + i\sin k\theta$

$$\text{So } z^{-k} = \frac{1}{\cos k\theta + i\sin k\theta} = \frac{\cos k\theta - i\sin k\theta}{(\cos k\theta + i\sin k\theta)(\cos k\theta - i\sin k\theta)}$$

$$= \frac{\cos k\theta - i\sin k\theta}{\cos^2 k\theta + \sin^2 k\theta} = \frac{\cos k\theta - i\sin k\theta}{1} = \cos k\theta - i\sin k\theta.$$

(b) From $z^k = \cos k\theta + i\sin k\theta$ and $z^{-k} = \cos k\theta - i\sin k\theta$,

adding gives $z^k + z^{-k} = 2\cos k\theta$, $\quad\therefore \cos k\theta = \frac{1}{2}(z^k + z^{-k})$

subtracting gives $z^k - z^{-k} = 2i\sin k\theta$,

$$\therefore \sin k\theta = \frac{1}{2i}(z^k - z^{-k}).\ \left[\text{or } \frac{i}{2}(z^{-k} - z^k)\right]$$

$$\cos 2\theta \sin^2\theta = \frac{1}{2}(z^2 + z^{-2})\frac{1}{(2i)^2}(z - z^{-1})^2$$
$$= -\frac{1}{8}\left(z^2 + \frac{1}{z^2}\right)\left(z - \frac{1}{z}\right)^2$$

(c) Expanding this last expression gives

$$-\frac{1}{8}(z^2 + z^{-2})(z^2 - 2 + z^{-2})$$
$$= -\frac{1}{8}(z^4 + 1 - 2z^2 - 2z^{-2} + 1 + z^{-4})$$
$$= -\frac{1}{8}(z^4 + z^{-4} + 2 - 2z^2 - 2z^{-2})$$
$$= -\frac{1}{8}(2\cos 4\theta + 2 - 4\cos 2\theta) = \frac{1}{2}\cos 2\theta - \frac{1}{4}\cos 4\theta - \frac{1}{4},$$

so $a = -\frac{1}{4}$, $b = \frac{1}{2}$, $c = -\frac{1}{4}$.

(2) (a) $z^4 = (\cos\theta + i\sin\theta)^4$

$$= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i^2\sin^2\theta) + 4\cos\theta(i^3\sin^3\theta) + i^4\sin^4\theta$$
$$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$$
$$= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$$

(b) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$

(c) Equating the real parts from (a) and (b):

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$\frac{\cos 4\theta}{\cos^2\theta} = \cos^2\theta - 6\sin^2\theta + \sin^4\theta\sec^2\theta$$
$$= \cos^2\theta - 6(1 - \cos^2\theta) + (1 - \cos^2\theta)^2\sec^2\theta$$
$$= 7\cos^2\theta - 6 + (1 - 2\cos^2\theta + \cos^4\theta)\sec^2\theta$$
$$= 7\cos^2\theta - 6 + (\sec^2\theta - 2 + \cos^2\theta)$$
$$= 8\cos^2\theta + \sec^2\theta - 8$$

$p = 8$, $q = 1$, $r = -8$.

(3) $z^3 = r^3(\cos 3\theta + i\sin 3\theta)$

$$\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^3 = \cos 2\pi + i\sin 2\pi$$

$a = 1$; $b = 0$

Method 1

$r^3(\cos 3\theta + i\sin 3\theta) = 8$

$r^3\cos 3\theta = 8$ and $r^3\sin 3\theta = 0$

$\Rightarrow r = 2;\ 3\theta = 0,\ 2\pi,\ 4\pi$

Roots are 2, $2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$, $2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$.

In Cartesian form: 2, $(-1 + i\sqrt{3})$, $(-1 - i\sqrt{3})$

Method 2

$$z^3 - 8 = 0$$

$$(z - 2)(z^2 + 2z + 4) = 0$$

$$(z - 2)((z + 1)^2 + (\sqrt{3})^2) = 0$$

so the roots are: 2, $(-1 + i\sqrt{3})$, $(-1 - i\sqrt{3})$

**Polynomial equations – page 50**

(1) $z^2(z + 3) = (1 + 4i - 4)(1 + 2i + 3)$

$= (-3 + 4i)(4 + 2i)$

$= -20 + 10i$.

$z^3 + 3z^2 - 5z + 25 = z^2(z + 3) - 5z + 25$

$= -20 + 10i - 5 - 10i + 25 = 0$

Another root is the conjugate of $z$, i.e. $1 - 2i$.

The corresponding quadratic factor is $(z - 1)^2 + 4 = z^2 - 2z + 5$.

$z^3 + 3z^2 - 5z + 25 = (z^2 - 2z + 5)(z + 5)$

so the third root is $z = -5$.

(2) Since $3 - i$ is a root, so is $3 + i$.

$(z - (3 - i))(z - (3 + i)) = (z - 3 + i)(z - 3 - i)$

$= (z - 3)^2 + 1$

$= z^2 - 6z + 10$

Hence $z^2 - 6z + 10$ is a factor of $2z^3 - 11z^2 + 14z + 10$.

By inspection or long division

$2z^3 - 11z^2 + 14z + 10 = (z^2 - 6z + 10)(2z + 1)$, so the roots are

$3 + i,\ 3 - i,\ -\frac{1}{2}$.

(3) $i^4 + 4i^3 + 3i^2 + 4i + 2 = 1 - 4i - 3 + 4i + 2 = 0$.

Since $i$ is a root, $-i$ must also be a root.

So, $(z - i)(z + i) = z^2 + 1$ is a factor.

$z^4 + 4z^3 + 3z^2 + 4z + 2 = (z^2 + 1)(z^2 + 4z + 2)$ by inspection or long division.

Solving $z^2 + 4z + 2 = 0$ gives $z = -2 \pm \sqrt{2}$.

The roots are $i,\ -i,\ -2 + \sqrt{2},\ -2 - \sqrt{2}$.

(4) Because $i - 1$ is a root, so is $-i - 1$, and

$(z - i + 1)(z + i + 1) = z^2 + 2z + 2$ is a factor of $f(z)$.

Dividing $z^4 - z^3 - 5z^2 - 8z - 2$ by $z^2 + 2z + 2$ gives

$z^4 - z^3 - 5z^2 - 8z - 2 = (z^2 + 2z + 2)(z^2 - 3z - 1)$, so other roots given by $z^2 - 3z - 1 = 0$ This is solved by the quadratic formula.

The roots are $-i - 1,\ i - 1,\ \frac{3}{2} \pm \frac{\sqrt{13}}{2}$.

**Geometric figures in the complex plane – page 51**

(1) Let $z = x + iy$.

$z + i = x + iy + i = x + (1 + y)i$

so, $|z + i| = \sqrt{x^2 + (1 + y)^2}$

hence $x^2 + (1 + y)^2 = 4$

which is a circle, centre $(0, -1)$, radius 2.

$|z + i| \leqslant 2$ consists of the circle and all points inside it.

## CHAPTER 8

**Arithmetic sequences – page 53**

(1) Let the first term be $a$ and the common difference be $d$:

$a = 3$ and $u_{10} = a + 9d = 3 + 9d = -15 \Rightarrow d = -2$.

$$S_{100} = \frac{n}{2}(2a + (n - 1)d) = \frac{100}{2}\{2 \times 3 + 99 \times (-2)\} = -9600.$$

or

$$S_{100} = an + \frac{n}{2}(n - 1)d = 300 + 50 \times 99 \times -2 = -9600$$

(2) $\displaystyle\sum_{r=1}^{n}(6 + 2r) = \sum_{r=1}^{n}6 + 2\sum_{r=1}^{n}r$

$\displaystyle = 6n + 2\left[\frac{1}{2}n(n + 1)\right] = 7n + n^2$

$\displaystyle\sum_{r=1}^{3k}(6 + 2r) = 7(3k) + (3k)^2 = 21k + 9k^2$.

$\displaystyle\sum_{r=k+1}^{3k}(6 + 2r) = \sum_{r=1}^{3k}(6 + 2r) - \sum_{r=1}^{k}(6 + 2r)$

$= 21k + 9k^2 - 7k - k^2$

$= 14k + 8k^2$

(3) $u_1 = S_1 = 8 - 1 = 7$

$u_2 = S_2 - S_1 = 12 - 7 = 5$

$u_3 = S_3 - S_2 = 15 - 12 = 3$

The sequence is arithmetic, with $a = 7$ and $d = -2$.

The $n$th term $u_n = a + (n - 1)d = 7 + (n - 1)(-2) = 9 - 2n$.

**Geometric sequences – page 55**

(1) If the first term is $a$ and the common ratio is $r$ then

$ar = -2,\ ar^3 = -\frac{2}{9}$

$\therefore \frac{ar^3}{ar} = \frac{-\frac{2}{9}}{-2} = \frac{1}{9}$

$\therefore r^2 = \frac{1}{9} \Rightarrow r = \pm\frac{1}{3}$.

As both satisfy $|r|<1$, each gives a sum to infinity.

$r=\frac{1}{3}$ gives $a=-6$ and sum to infinity $\frac{-6}{1-\frac{1}{3}}=-9$.

$r=-\frac{1}{3}$ gives $a=6$ and sum to infinity $\frac{6}{1-\left(-\frac{1}{3}\right)}=\frac{6}{\frac{4}{3}}=\frac{9}{2}$.

(2) (a) $r=\dfrac{\frac{x(x+1)^2}{(x-2)^2}}{\frac{x(x+1)}{(x-2)}}=\frac{(x+1)}{(x-2)}$

$u_n=ar^{n-1}$

$=\frac{x(x+1)^n}{(x-2)^n}$

(b) $S_n=\frac{a(r^n-1)}{r-1}$

$$=\frac{x(x+1)\left(\frac{(x+1)^n}{(x-2)^n}-1\right)}{(x-2)\left(\frac{x+1}{x-2}-1\right)}$$

$$=\frac{1}{3}x(x+1)\left(\frac{(x+1)^n}{(x-2)^n}-1\right)$$

(c) For a sum to infinity, $-1<r<1$, i.e. $r^2<1$

$$\frac{(x+1)^2}{(x-2)^2}<1$$

$$x^2+2x+1<x^2-4x+4$$

$$6x<3$$

i.e. $x<\frac{1}{2}$

$$S=\frac{a}{1-r}=\frac{x(x+1)}{(x-2)\left(1-\frac{x+1}{x-2}\right)}$$

$$=\frac{-x(x+1)}{3}$$

(3) $a_j=p^j \Rightarrow S_k=p+p^2+\dots+p^k=\frac{p(p^k-1)}{p-1}$

$S_n=\frac{p(p^n-1)}{p-1}$

$S_{2n}=\frac{p(p^{2n}-1)}{p-1}$

$\frac{p(p^{2n}-1)}{p-1}=\frac{65p(p^n-1)}{p-1}$

$(p^n+1)(p^n-1)=65(p^n-1)$

$p^n+1=65$

$\Rightarrow p^n=64$

$a_2=p^2 \Rightarrow a_3=p^3$ but $a_3=2p$ so $p^3=2p$

$\Rightarrow p^2=2 \Rightarrow p=\sqrt{2}$ since $p>0$.

$p^n=64=2^6=(\sqrt{2}^{12})^{12}$

$n=12$

**Maclaurin series – page 58**

(1) $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots$

$=x-\frac{x^3}{6}+\frac{x^5}{120}-\dots$

so, replacing $x$ by $3x$,

$\sin 3x=3x-\frac{27x^3}{6}+\frac{243x^5}{120}-\cdots$

$=3x-\frac{9x^3}{2}+\frac{81x^5}{40}-\cdots$

$(1+x)\sin 3x=(1+x)\left(3x-\frac{9x^3}{2}+\frac{81x^5}{40}-\cdots\right)$

$=3x-\frac{9x^3}{2}+\frac{81x^5}{40}-\cdots+3x^2-\frac{9x^4}{2}+\cdots$

$=3x+3x^2-\frac{9}{2}x^3-\frac{9}{2}x^4+\frac{81}{40}x^5+\cdots$

(2) Let $f(x)=\ln(3-x)$: $f(0)=\ln 3$;

$f'(x)=-\frac{1}{3-x}$, $f'(0)=-\frac{1}{3}$;

$f''(x)=-\frac{1}{(3-x)^2}$, $f''(0)=-\frac{1}{9}$

So $f(x)=\ln 3-\frac{x}{3}-\frac{1}{2}\frac{x^2}{9}+\dots=\ln 3-\frac{x}{3}-\frac{x^2}{18}+\cdots$

$x^2\ln(3-x)=x^2\ln 3-\frac{x^3}{3}-\frac{x^4}{18}+\cdots$

and $x^2\ln(3+x)=x^2\ln 3+\frac{x^3}{3}-\frac{x^4}{18}+\cdots$

$x^2\ln(9-x^2)=x^2\ln(3-x)+x^2\ln(3+x)=2x^2\ln 3-\frac{x^4}{9}+\cdots$

(3) $\frac{x^2+8x+11}{(x+1)(x+3)^2}=\frac{A}{x+1}+\frac{B}{x+3}+\frac{C}{(x+3)^2}$

$x^2+8x+11=A(x+3)^2+B(x+1)(x+3)+C(x+1)$

$x=-1$ gives $4=4A$, $A=1$;

$x=-3$ gives $-4=-2C$, $C=2$;

$x=-2$ gives $-1=A-B-C$, $-1=1-B+2$, $B=0$;

or equating the coefficient of $x^2$ on both sides:
$1=A+B$, so $B=0$.

$f(x)=\frac{x^2+8x+11}{(x+1)(x+3)^2}=\frac{1}{x+1}+\frac{2}{(x+3)^2}$; $f(0)=\frac{11}{9}$

$f'(x)=-\frac{1}{(x+1)^2}-\frac{4}{(x+3)^3}$; $f'(0)=\frac{-31}{27}$

$f''(x)=\frac{2}{(x+1)^3}+\frac{12}{(x+3)^4};\quad f''(0)=\frac{58}{27}.$

$f(x)=\frac{11}{9}-\frac{31}{27}x+\frac{58}{27}\frac{x^2}{2}-\cdots\quad=\frac{11}{9}-\frac{31}{27}x+\frac{29}{27}x^2-\cdots$

The series for $\frac{1}{1+x}$ converges for $|x|<1$.

Writing $\frac{1}{x+3}=\frac{1}{3}\left(1+\frac{x}{3}\right)^{-1}$ shows that the series for $\frac{1}{3+x}$ converges for $|x|<3$.

For both terms to converge, we need $|x|<1$.

(4) $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$

$e^{x^2}=1+x^2+\frac{x^4}{2!}+\cdots$

$e^{x+x^2}=e^xe^{x^2}=\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\cdots\right)\left(1+x^2+\frac{x^4}{2}+\cdots\right)$

$=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+x^2+x^3+\frac{x^4}{2}+\frac{x^4}{2}+\cdots$

$=1+x+\frac{3}{2}x^2+\frac{7}{6}x^3+\frac{25}{24}x^4+\cdots$

**Iterative schemes – page 59**

(1) Let the fixed point be $L$, then

$$L=\ln(1+\tfrac{1}{2}e^L)$$
$$\Rightarrow e^L=1+\tfrac{1}{2}e^L.$$

So $e^L=2\quad\Rightarrow\quad L=\ln 2$.

(2) If $\lambda$ is a fixed point, then

$\lambda=\frac{1}{2}\left\{\lambda+\frac{7}{\lambda}\right\}$

$2\lambda=\lambda+\frac{7}{\lambda}$

$\lambda=\frac{7}{\lambda}$

$\lambda^2=7$.

The fixed points are $\sqrt{7}$ and $-\sqrt{7}$.

## CHAPTER 9

**Basic skills information – page 61**

(1) $\boldsymbol{v}\times\boldsymbol{w}=\begin{vmatrix} i & j & k\\ 2 & -1 & 2\\ -1 & 0 & 2\end{vmatrix}=\boldsymbol{i}\begin{vmatrix}-1 & 2\\ 0 & 2\end{vmatrix}-\boldsymbol{j}\begin{vmatrix}2 & 2\\ -1 & 2\end{vmatrix}+\boldsymbol{k}\begin{vmatrix}2 & -1\\ -1 & 0\end{vmatrix}$

$=-2\boldsymbol{i}-6\boldsymbol{j}-\boldsymbol{k}$

$(\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}).(-2\boldsymbol{i}-6\boldsymbol{j}-\boldsymbol{k})=1(-2)+1(-6)+(-1)(-1)=-7.$

**Equations of lines – page 63**

(1) (a) Parametric equations for the lines are:

$L_1$: $x=-3-2s,\ y=1+s,\ z=5+3s$

$L_2$: $x=1+t,\ y=1-t,\ z=3-t$

Equating the corresponding $x$ and $y$ coordinates and solving for $s$ and $t$ gives

$x$: $-3-2s=1+t$, $y$: $1+s=1-t$, $\Rightarrow\ s=-4, t=4$.

For $L_1$ this gives $z=-7$, and for $L_2$ $z=-1$. Hence the lines do not intersect.

(b) A vector perpendicular to the directions of $L_1$ and $L_2$ is given

by $\begin{vmatrix} i & j & k\\ -2 & 1 & 3\\ 1 & -1 & -1\end{vmatrix}=2\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$,

so $L_3$ has parametric equations $x=-1+2s,\ y=s,\ z=2+s$.

(c) The lines $L_3$ and $L_2$ intersect if the equations

$-1+2s=1+t,\ s=1-t,\ 2+s=3-t$ have a unique solution.

Solving the first two equations gives $s=1$, $t=0$, and these satisfy the third equation also, so the lines intersect. $Q$ is the point (1, 1, 3) ($t=0$ for $L_2$ or $s=1$ for $L_3$)

(d) $P(-1,0,2)$: for $L_1$ $x=-3-2s$, setting $x=-1$, gives

$-1=-3-2s\Rightarrow s=-1$.

$y=1+s\Rightarrow y=0,\ z=5+3s\Rightarrow z=2$ which shows

that P lies on $L_1$.

$P$ lies on $L_1$, $Q$ lies on $L_2$, and the line $PQ$ is perpendicular to both lines.

**Intersections of lines and planes – page 67**

(1) (a) A normal vector to the plane is given by the direction vector of the line: $\boldsymbol{n}=\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$

$\boldsymbol{n}=\begin{pmatrix}1\\2\\1\end{pmatrix},\quad \boldsymbol{u}=\begin{pmatrix}2\\1\\-1\end{pmatrix},\quad \boldsymbol{n}.\boldsymbol{u}=\begin{pmatrix}1\\2\\1\end{pmatrix}.\begin{pmatrix}2\\1\\-1\end{pmatrix}=2+2-1=3$

An equation for the plane is $x+2y+z=3$.

or

An equation for the plane is $1.(x-2)+2.(y-1)+1.(z+1)=0$, i.e. $x+2y+z=3$.

Parametric equations for $L$ are: $x=\lambda-3,\ y=2\lambda,\ z=\lambda+12$

(b) The plane and $L$ meet when

$(\lambda-3)+2(2\lambda)+(\lambda+12)=3\ \Rightarrow\ 6\lambda+9=3\ \Rightarrow\ \lambda=-1$.

Setting $\lambda=-1$ in the parametric equations for $L$ gives $B(-4,-2,11)$.

(c) $AB^2=(2+4)^2+(1+2)^2+(1+11)^2=189$,

so $AB=\sqrt{189}=3\sqrt{21}$.

This gives the shortest distance from $A$ to $L$ because $B$ is the foot of the perpendicular from $A$ onto $L$.

# NTT Answers

**Angles between lines and planes – page 71**

(1) (a) Interchange the first two equations:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & -1 & 1 & 8 \\ -1 & 3 & 2 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|cl} 1 & 2 & 1 & 3 & \\ 0 & -5 & -1 & 2 & r_2-2r_1 \\ 0 & 5 & 3 & 4 & r_3+r_1 \end{array}\right)$$

$$\left(\begin{array}{ccc|cl} 1 & 2 & 1 & 3 & \\ 0 & 5 & 1 & -2 & -r_2 \\ 0 & 0 & 2 & 6 & r_3+r_2 \end{array}\right)$$

Solving gives $z=3,\ y=-1,\ x=2.$

(b) Setting $y=-t$ in the two equations gives

$2x+t+z=8,\ x-2t+z=3.$

$$\begin{aligned} 2x+z&=8-t \\ x+z&=3+2t \\ \hline x=5-3t,&\quad z=5t-2 \end{aligned}$$

Thus, $x=5-3t,\quad y=-t,\quad z=5t-2.$

(c) We need the acute angle between the line and the plane.

Firstly, we find the acute angle between $\boldsymbol{d}=\begin{pmatrix}-3\\-1\\5\end{pmatrix}$, the direction vector of the line, and $\boldsymbol{n}=\begin{pmatrix}-1\\3\\2\end{pmatrix}$, the normal vector to the plane.

$\boldsymbol{d}.\boldsymbol{n}=(-3)(-1)+(-1)3+(5)(2)=10,\ |\boldsymbol{d}|=\sqrt{35},\ |\boldsymbol{n}|=\sqrt{14}.$

$\cos\theta=\dfrac{10}{\sqrt{35}\sqrt{14}}$, giving $\theta=63\cdot1°$ to three significant figures.

Thus the angle between the line and the plane is $90°-63\cdot1°=26\cdot9°$

(2) (a) $\overrightarrow{AB}=2\boldsymbol{i}-\boldsymbol{k},\ \overrightarrow{AC}=\boldsymbol{i}-\boldsymbol{j}-3\boldsymbol{k}$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 2 & 0 & -1\\ 1 & -1 & -3\end{vmatrix}$$

$$=-\boldsymbol{i}+5\boldsymbol{j}-2\boldsymbol{k}$$

Equation of $\pi_1$ is of the form $-x+5y-2z=d$

(1, 1, 0) is on $\pi_1$, so setting $x=y=1,\ z=0$ gives $d=4$.

An equation for $\pi_1$ is $-x+5y-2z=4$.

(b) Normals for $\pi_1$ and $\pi_2$ are $-\boldsymbol{i}+5\boldsymbol{j}-2\boldsymbol{k}$ and $\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$.

The angle between the planes is the angle between their normals, given by

$$\cos^{-1}\left(\frac{-1+10-2}{\sqrt{30}\sqrt{6}}\right)=\cos^{-1}\frac{7}{6\sqrt{5}}\approx 58.6°.$$

(3) (a) $2x+y=3$

(b) $a=3,\ b=7$

$x=0+2t;\ y=3-4t;\ z=7-10t$

(c) $\cos\theta=\left|\dfrac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\right|=\dfrac{5}{\sqrt{55}}$

$\left\{\text{or } \sin\theta=\left|\dfrac{\boldsymbol{a}\times\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\right|=\sqrt{\dfrac{6}{11}}\right\}$

hence $\theta\approx 47.6°$

## CHAPTER 10

**First-order equations – page 75**

(1) $\dfrac{dy}{dx}-\dfrac{2}{x}y=x^2e^{2x};$

integrating factor is $e^{\int\frac{-2}{x}dx}=e^{-2\ln x}=e^{\ln x^{-2}}=x^{-2}=\dfrac{1}{x^2}.$

So $\dfrac{1}{x^2}\dfrac{dy}{dx}-\dfrac{2}{x^3}y=e^{2x}$

$\Rightarrow\dfrac{d}{dx}\left(\dfrac{y}{x^2}\right)=e^{2x}$

$\Rightarrow\dfrac{y}{x^2}=\int e^{2x}dx$

$\Rightarrow\dfrac{y}{x^2}=\dfrac{1}{2}e^{2x}+c$

so $y=\dfrac{1}{2}x^2e^{2x}+cx^2.$

$y(1)=0$ gives $0=\frac{1}{2}e^2+c$, so $c=-\frac{1}{2}e^2$

and $y=\dfrac{1}{2}x^2e^{2x}-\dfrac{1}{2}e^2x^2\Rightarrow y=\dfrac{1}{2}x^2\left(e^{2x}-e^2\right).$

(2) (a) $G=25k(1-e^{-t}/25)$

(b) $k\approx 0\cdot132$

(c) When $t=10$, $G\approx 1\cdot09$, the claim appears to be justified.

(d) As $t\to\infty$, $G\to 25k\approx 3\cdot3$ metres, so limit is 3·3 metres.

(3) (a) $\dfrac{dy}{dx}-\dfrac{3}{x}y=x^3$

Integrating factor is

$\exp\left(\int\dfrac{-3}{x}dx\right)=\exp(-3\ln x)=\exp(\ln(x^{-3}))=x^{-3}$

$\dfrac{1}{x^3}\dfrac{dy}{dx}-\dfrac{3}{x^4}y=1\quad\Rightarrow\quad\dfrac{d}{dx}\left(\dfrac{y}{x^3}\right)=1$

so $\dfrac{y}{x^3}=x+C\quad\Rightarrow\quad y=x^4+Cx^3$ where $C$ is an arbitrary constant.

The general solution is $y=x^4+Cx^3$.

$y=2$ when $x=1$ gives $C=1$.

The particular solution is $y=x^4+x^3$.

(b) $y\dfrac{dy}{dx}=x^4+3x$

so $\int y\,dy=\int(x^4+3x)\,dx$

$\frac{y^2}{2}=\frac{x^5}{5}+\frac{3x^2}{2}+D$ where $D$ is an arbitrary constant.

$y=2$ when $x=1$ gives $2=\frac{1}{5}+\frac{3}{2}+D \Rightarrow D=\frac{3}{10}$

Hence the particular solution is $y=\sqrt{\frac{2}{5}x^5+3x^2+\frac{3}{5}}$.

**Second-order equations with constant coefficients – page 79**

(1) The auxiliary equation is

$m^2+6m+9=0$
$(m+3)^2=0$ giving repeated root $m=-3$.

The general solution is $y=(A+Bx)e^{-3x}$;

$y=2$ when $x=0$ gives $A=2$.

$y=(2+Bx)e^{-3x} \Rightarrow \frac{dy}{dx}=-3(2+Bx)e^{-3x}+Be^{-3x}$,

so $\frac{dy}{dx}=-3$ when $x=0$ gives $B=3$.

The solution is $y=(2+3x)e^{-3x}$.

(2) Auxiliary equation is $m^2+4m-5=0 \Rightarrow (m-1)(m+5)=0$.

Roots are $m=1$, $m=-5$; complementary function:
$y=Ae^{x}+Be^{-5x}$.

For a particular solution try $y=Ce^{2x}+D$;

$\frac{dy}{dx}=2Ce^{2x}$, $\frac{d^2y}{dx^2}=4Ce^{2x}$.

Substituting into the full equation gives
$4Ce^{2x}+8Ce^{2x}-5Ce^{2x}-5D=7e^{2x}+10$
i.e. $7Ce^{2x}-5D=7e^{2x}+10$; equating coefficients gives $C=1$ and $D=-2$.

The general solution of the full equation is

$y=Ae^{x}+Be^{-5x}+e^{2x}-2$.

$y(0)=1$ gives $A+B=2$; $y'(0)=10$ gives $A-5B=8$.

Solving gives $A=3$, $B=-1$.

and the particular solution is $y=3e^{x}-e^{-5x}+e^{2x}-2$.

(3) Auxiliary equation is

$m^2+4m+8=0 \Rightarrow (m+2)^2+4=0 \Rightarrow m=-2\pm 2i$.

The complementary function is $y=e^{-2x}(A\cos 2x+B\sin 2x)$.

For a particular solution try $y=Cx^2+Dx+E$;
$y'=2Cx+D$, $y''=2C$.

Substituting into the full equation:

$2C+4(2Cx+D)+8(Cx^2+Dx+E)=8x^2+16x+6$

i.e. $8Cx^2+(8C+8D)x+2C+4D+8E=8x^2+16x+6$.

Equating coefficients gives $C=1$, $8+8D=16$, $6+8E=6$, so $C=1$, $D=1$, $E=0$.

The general solution of the full equation is

$y=e^{-2x}(A\cos 2x+B\sin 2x)+x^2+x$.

(4) Auxiliary equation is

$m^2+2m+5=0 \Rightarrow (m+1)^2+4=0 \Rightarrow m=-1\pm 2i$.

The general solution is $y=e^{-x}(A\cos 2x+B\sin 2x)$.

$y=1$ when $x=0$ gives $A=1$;

$y=2$ when $x=\frac{\pi}{4}$ gives $2=e^{-\pi/4}(0+B) \Rightarrow B=2e^{\pi/4}$.

The solution is $y=e^{-x}(\cos 2x+2e^{\pi/4}\sin 2x)$.

## CHAPTER 11

**Numbers and direct proof – page 81**

(1) $L.H.S=\frac{1}{\cos^2 x}$

$=\frac{\sin^2 x+\cos^2 x}{\cos^2 x}$

$=\frac{\sin^2 x}{\cos^2 x}+\frac{\cos^2 x}{\cos^2 x}$

$=\tan^2 x+1$

$=R.H.S$ as required

or

$R.H.S=1+\tan^2 x$

$=1+\frac{\sin^2 x}{\cos^2 x}$

$=\frac{\cos^2 x}{\cos^2 x}+\frac{\sin^2 x}{\cos^2 x}$

$=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}$

$=\frac{1}{\cos^2 x}$

$=\sec^2 x$

$=L.H.S$ as required

**Proof by contradiction – page 83**

(1) Let $x$ be irrational, and assume that $\frac{1}{x}$ is rational.

Then $\frac{1}{x}=\frac{p}{q}$, where $p$ and $q$ are integers.

But $\frac{1}{x}=\frac{p}{q} \Rightarrow x=\frac{q}{p}$, which means that $x$ is rational.

This contradicts the original assumption, therefore

$x$ irrational $\Rightarrow \frac{1}{x}$ is irrational.

**Further proofs – page 85**

(1) A. Let $p=2m$, (even)

$q=2n+1$, (odd)

so $p^3+q^2=(2m)^3+(2n+1)^2=8m^3+4n^2+4n+1$
$=2(4m^3+2n^2+2n)+1$.

This is odd, being of the form $2k+1$, where $k$ is an integer, so $A$ is true.

B. The counter example $m=3$ shows that B is false.

$m^2$ is divisible by 9, but $m$ is not divisible by 9

(2)

'If' part:

$100a + 10b + c = 99a + 9b + a + b + c$, so if $a + b + c = 9k$, where $k$ is an integer, $100a + 10b + c = 99a + 9b + 9k$, which is divisible by 9.

'Only if" part:

if $100a + 10b + c = 9m$, where $m$ is an integer, then $100a + 10b + c = 99a + 9b + a + b + c$ gives $9m = 99a + 9b + a + b + c$, and so $a + b + c = 9m - 99a - 9b$ is divisible by 9.

(3) Statement A is true: $p(n) = n(n+1)$ and one of $n$ and $n+1$ must be even.

Statement B is false: a counterexample is $n = 1$, $p(1) = 1^2 + 1 = 2$.

**Proofs by induction – page 87**

(1) Prove by induction that $\sum_{r=1}^{n}(6r^2 + 4r) = n(n+1)(2n+3)$.

When $n = 1$

L.H.S: $6(1)^2 + 4(1) = 10$, R.H.S: $1 \times 2 \times 5 = 10$

so result true for $n = 1$.

Assume that for some positive integer $k$

$\sum_{r=1}^{k}(6r^2 + 4r) = k(k+1)(2k+3)$.

Then $\sum_{r=1}^{k+1}(6r^2 + 4r) = \sum_{r=1}^{k}(6r^2 + 4r) + \left[6(k+1)^2 + 4(k+1)\right]$

$= k(k+1)(2k+3) + 6(k+1)^2 + 4(k+1)$

$= (k+1)\left[k(2k+3) + 6(k+1) + 4\right]$

$= (k+1)\left[2k^2 + 9k + 10\right]$

$= (k+1)(k+2)(2k+5)$

$= (k+1)((k+1)+1)(2(k+1)+3)$

which is the result for $k$ with $k$ replaced by $k + 1$.

So if true for $n = k$, then true for $n = k+1$. But true for $n = 1$, then by induction, true for all integers $n \geq 1$.

(2) Prove by induction on $n$ that for $x > 0$

$(1+x)^n \geq 1 + nx + \frac{1}{2}n(n-1)x^2$

for all positive integers $n$.

When $n = 1$

L.H.S: $(1+x)^1 = (1+x)$, R.H.S: $1 + x + \frac{1}{2}1.(1-1)x^2 = 1 + x$

L.H.S ≥ R.H.S, so true for $n = 1$.

Assume that for some positive integer $k$:

$(1+x)^k \geq 1 + kx + \frac{1}{2}k(k-1)x^2$

Then

$(1+x)^{k+1} = (1+x)(1+x)^k \geq (1+x)\left(1 + kx + \frac{1}{2}k(k-1)x^2\right)$

because $1 + x > 0$.

So $(1+x)^{k+1} \geq (1+x)\left(1 + kx + \frac{1}{2}k(k-1)x^2\right)$

$= 1 + kx + \frac{1}{2}k(k-1)x^2 + x + kx^2 + \frac{1}{2}k(k-1)x^3$

$= 1 + (k+1)x + \left[\frac{1}{2}k(k-1) + k\right]x^2 + \frac{1}{2}k(k-1)x^3$

$\geq 1 + (k+1)x + \left[\frac{1}{2}k(k-1) + k\right]x^2$ since $k \geq 1$ and $x > 0 \Rightarrow \frac{1}{2}k(k-1)x^3 \geq 0$

$= 1 + (k+1)x + \frac{1}{2}(k+1)kx^2$,

which is the result for $k$ with $k$ replaced by $k + 1$. So if true for $k$, then true for $k + 1$. But true for $n = 1$, hence by induction true for all integers $n \geq 1$.

(3) $(AB)^{-1} = B^{-1}A^{-1}$. [1]

When $n = 1$

L.H.S: $\left(A^1\right)^{-1} = A^{-1}$, R.H.S: $\left(A^{-1}\right)^1 = A^{-1}$, so the result is true for $n = 1$. [2]

Assume that for some integer $k$, $\left(A^k\right)^{-1} = \left(A^{-1}\right)^k$ [3]

and consider $\left(A^{k+1}\right)^{-1}$.

$A^{k+1} = A^k A^1$

$\left(A^{k+1}\right)^{-1} = \left(A^k A^1\right)^{-1}$

$= \left(A^1\right)^{-1}\left(A^k\right)^{-1}$ from [1]

$= A^{-1}\left(A^{-1}\right)^k$ from [2] and [3]

$= \left(A^{-1}\right)^{k+1}$

or

[$A^{k+1} = AB$, where $B = A^k$,
so $(A^{k+1})^{-1} = (AB)^{-1} = B^{-1}A^{-1} = (A^k)^{-1}A^{-1}$.

By hypothesis, $(A^k)^{-1} = (A^{-1})^k$, so
$(A^{k+1})^{-1} = (A^k)^{-1}A^{-1} = (A^{-1})^k A^{-1} = (A^{-1})^{k+1}$].

This proves that the inverse of $A^{k+1}$ is $(A^{-1})^{k+1}$.

So true for $n = k$ implies true for $n = k + 1$, and because the result holds for $n = 1$, the result, by induction, is true for all integers $n \geq 1$.

(4) $\frac{d}{dx}(xe^x) = 1.e^x + xe^x = (x+1)e^x$

LHS: $\mathbf{1.e^x + xe^x}$, RHS: $\mathbf{(x + 1)e^x}$

hence the result holds for $n = 1$.

Assume that, for some positive integer $k$,

$\frac{d^k}{dx^k}(xe^x) = (x+k)e^x$.

Then $\frac{d^{k+1}}{dx^{k+1}}(xe^x) = \frac{d}{dx}\left(\frac{d^k}{dx^k}(xe^x)\right) = \frac{d}{dx}((x+k)e^x)$ by above assumption.

$\frac{d}{dx}((x+k)e^x) = e^x + (x+k)e^x = (x+(k+1))e^x$

So, if the result is true for $n = k$, it is true for $n = k + 1$. But the result is true for $n = 1$, hence true for all integers $n \geqslant 1$.

**Euclidean algorithm – page 89**

(1) $29400 = 4 \times 6860 + 1960$

$6860 = 3 \times 1960 + 980$

$1960 = 2 \times 980 + 0$

Hence the gcd of 29400 and 6860 is 980.

$980 = 6860 - 3 \times 1960$

$\quad = 6860 - 3(29400 - 4 \times 6860)$

$\quad = 13 \times 6860 - 3 \times 29400$

i.e. $a = -3 \quad b = 13$

(2) $426_7 = 4 \times 7^2 + 2 \times 7 + 6 \times 7^0$

$\quad = 196 + 14 + 6$

$\quad = 216_{10}$

$216 = 1 \times 125 + 91$

$91 = 3 \times 25 + 16$

$16 = 3 \times 5 + 1$

$1 = 1 \times 1$

Hence $426_7 = 216_{10} = 1331_5$

or

$$\frac{216}{5} = 43 + \frac{1}{5}, \quad \frac{216}{5^2} = 8 + \frac{3}{5} + \frac{1}{5^2}, \quad \frac{216}{5^3} = 1 + \frac{3}{5} + \frac{3}{5^2} + \frac{1}{5^3}.$$

Hence $426_7 = 216_{10} = 5^3 + 3.5^2 + 3.5 + 1 = 1331_5$.

(3) $231 = 13 \times 17 + 10$

$17 = 1 \times 10 + 7$

$10 = 1 \times 7 + 3$

$7 = 2 \times 3 + 1$.

Hence (231, 17) = 1.

$1 = 7 - 2 \times 3$

$\quad = 7 - 2(10 - 7) = 3 \times 7 - 2 \times 10$

$\quad = 3 \times (17 - 10) - 2 \times 10 = 3 \times 17 - 5 \times 10$

$\quad = 3 \times 17 - 5(231 - 13 \times 17)$

$\quad = 68 \times 17 - 5 \times 231$

So, $x = -5$ and $y = 68$.