VECTORS

Vectors are used in physics and mechanics to deal with forces such as gravity, velocity, and weight.

A large part of the vectors work in Higher Maths is the study of vectors represented by directed lines in 3-dimensional space, with the axes at right angles to each other. This makes it a very useful topic for solving problems on shapes in three dimensions.

The results also hold in two dimensions and the calculations would be just the same but without *z*-coordinates and *z*-components – but you can expect the questions in the exam to be about 3 dimensions. Generally problems in 2 dimensions are dealt with by the geometry in The Straight Line chapter.

Vectors questions are usually done well in the Higher exam. Know the topic well and you will have a whole bunch of marks you can be sure of getting fairly quickly and easily.

VECTORS 1

DON'T FORGET

Don't be casual about how you write vectors. Expect to lose marks if you confuse the ways of writing coordinates and components in the exam. Remember coordinates label a point, whereas components give directions for a move from one place to another.

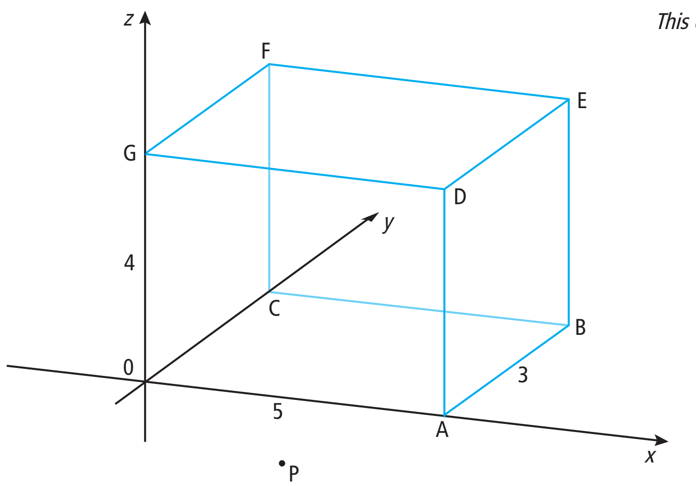
DON'T FORGET

You will usually have to interpret a skeleton diagram for vectors in the exam. Not everyone finds them easy– you may have to work hard to understand where all the points and lines actually are in relation to each other. If this is the case for you, make sure you study every example you can of a 3-d vector diagram in past exam papers.

WORKING WITH COORDINATES AND COMPONENTS

E (5, 3, 4) is a point in 3-dimensional space (see diagram below) $\mathbf{e} = \begin{pmatrix}5\\3\\4\end{pmatrix}$ is a vector and must be written with the components vertical.

In the diagram below the position vector **e** is the directed line segment OE, and gives the components of the move from O, the origin, to E. **e** is the *position vector* of E because it fixes its position. There can only be one point E. However there can be any number of representations of the vector **e** because every vector with the same components is an identical move or journey, but starting in a different place in space.



This diagram is used for most of the examples in the text

The cuboid pictured has been drawn on a coordinate diagram with *x*, *y* and *z*-axes in 3 dimensions at right angles to each other. Point P lies under the cuboid.

E is the point (5, 3, 4), so OA = 5 units, OC = 3 and OG = 4 in length.

Important and very useful result, $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$. It's very easy to verify for particular cases of vectors and points on coordinate diagrams.

cont

WORKING WITH COORDINATES AND COMPONENTS cont

Example 1

$\overrightarrow{PF} = \begin{pmatrix}-3\\1\\10\end{pmatrix}$ Find the coordinates of P.

F(0, 3, 4) is easily deduced from the lengths in the diagram.

Since $\overrightarrow{PF} = \mathbf{f} - \mathbf{p}$, and calling P (*x*, *y*, *z*), substitution gives us $\begin{pmatrix}-3\\1\\10\end{pmatrix} = \begin{pmatrix}0\\3\\4\end{pmatrix} - \begin{pmatrix}x\\y\\z\end{pmatrix}$

This vector equation is easily rearranged and solved to give

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\2\\-6\end{pmatrix}$ so P is the point (3, 2, –6)

USING VECTOR PATHWAYS

This is basic to understanding vectors. A simple case is $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ above.

The vector sum $\overrightarrow{FE} + \overrightarrow{PF} - \overrightarrow{PE}$

rearranged

$= \overrightarrow{FE} + \overrightarrow{PF} + \overrightarrow{EP}$ ($\overrightarrow{EP}$ is the negative of $\overrightarrow{PE}$ – same magnitude, opposite direction)

$= \overrightarrow{FE} + \overrightarrow{EP} + \overrightarrow{PF}$ (vectors can be added in any order)

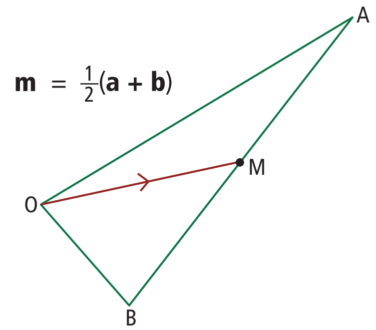
This gives a continuous route from F back to F – resulting in no change of position at all!

We have the components of $\overrightarrow{PF}$ above and it is easy to work out the components of $\overrightarrow{FE}$ and $\overrightarrow{EP}$.

Using components, $\overrightarrow{FE} + \overrightarrow{EP} + \overrightarrow{PF} = \begin{pmatrix}5\\0\\0\end{pmatrix} + \begin{pmatrix}-2\\-1\\-10\end{pmatrix} + \begin{pmatrix}-3\\1\\10\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

The result is the zero vector – unsurprising as we finished where we started.

Another illustration of vector paths is the result for finding the midpoint of a line. If M is the midpoint of $\overrightarrow{EP}$, then the diagram here shows



$\mathbf{m} = \overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$ – start at O, move to A, then halfway along $\overrightarrow{AB}$

$= \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a})$

$= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$ remember $\frac{1}{2}\mathbf{a}$ is $\mathbf{a}$ with all components halved

$= \frac{1}{2}(\mathbf{a} + \mathbf{b})$

LET'S THINK ABOUT THIS

Although we introduce O to do the working, $\mathbf{m} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ notice that the result doesn't depend on where O is but only on the positions of A and B in relation to each other.

You need to introduce O to establish $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ too. Can you write out the working for this?

VECTORS 2

MAGNITUDE OF A VECTOR AND THE DISTANCE FORMULA

DON'T FORGET

Don't go on to give an approximate answer by evaluating the square root, unless it is clear from the question that this is wanted, or is helpful for what you have to do next.

Example 2

Find $\overrightarrow{PF}$, the magnitude or length of $\overrightarrow{PF}$

Magnitude is calculated using Pythagoras Theorem, extended to three dimensions

$|\overrightarrow{PF}| = \sqrt{(-3)^2 + 1^2 + 10^2} = \sqrt{110}$

In general, the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

which you will recognise as the Distance Formula, again extended to three dimensions. If you substitute the coordinates of P and F you will find you are soon working out exactly the same as in the first calculation above.

UNIT VECTORS

These have length 1. Examples are **i**, **j** and **k** which take the directions of the three axes

$\mathbf{i} = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \mathbf{j} = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad \mathbf{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ so that $\mathbf{e} = \begin{pmatrix}5\\3\\4\end{pmatrix} = 5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

Example 3

Find a unit vector, **u**, parallel to $\overrightarrow{PF}$

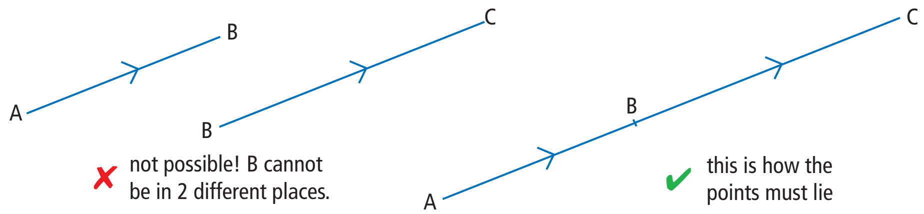
u will be a scalar multiple of $\overrightarrow{PF}$ (in order to be parallel)

$\overrightarrow{PF}$ has length $\sqrt{110}$ and u has length 1. Dividing each component of $\overrightarrow{PF}$ by $\sqrt{110}$ will make its length 1.

$\mathbf{u} = \frac{1}{\sqrt{110}}\begin{pmatrix}3\\2\\-6\end{pmatrix}$ (no need to evaluate further unless the question requires it)

COLLINEAR POINTS AND THE SECTION FORMULA

If you found that two lines had the same gradient you would know they were parallel. If you also knew that a particular point lay on both lines, then the two lines would no longer be separate lines but would both be parts of the same straight line.



DON'T FORGET

parallel lines + common point = collinearity

The vector work in this section does the same thing, but can be used in three dimensions (as well as two, should you want to). To show lines are parallel using vectors, show that one is a scalar multiple of the other (as shown below) If the lines/vectors also have a point in common, then they are one line and all the points lie on the same line, ie are collinear.

cont

COLLINEAR POINTS AND THE SECTION FORMULA cont

Example 4

Show that A(3, –1, 5), B(6, 0, 3) and C(15, 3, –3) are collinear points and draw a diagram to show their relative positions.

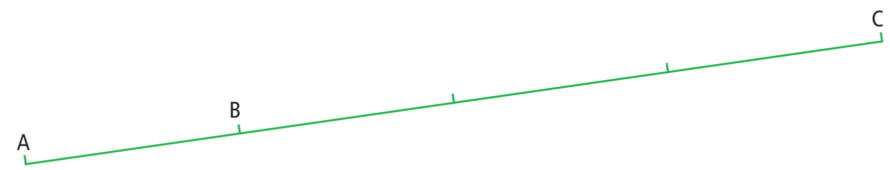
Choose any two directed line segments using these points.

Choosing $\overrightarrow{AB}$ and $\overrightarrow{AC}$ –

$\overrightarrow{AB} = \begin{pmatrix}6-3\\0+1\\3-5\end{pmatrix} = \begin{pmatrix}3\\1\\-2\end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix}15-3\\3+1\\-3-5\end{pmatrix} = \begin{pmatrix}12\\4\\-8\end{pmatrix} = 4\begin{pmatrix}3\\1\\-2\end{pmatrix}$

$\overrightarrow{AC} = 4\overrightarrow{AB}$ (ie $\overrightarrow{AC}$ is a scalar multiple of $\overrightarrow{AB}$) is easily spotted once the components are calculated, and since A is a common point, A, B and C are collinear points.

Next draw a line – the direction is irrelevant – and mark the points A, B and C on it so that the AC is four times the length of AB.



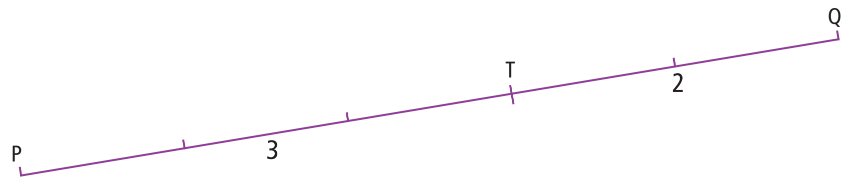
DON'T FORGET

Notice that AB : BC = 1 : 3 while AB : AC = 1 : 4 Diagrams like this one above will help you not to make mistakes based on exactly which line segments are which. It's wise to draw a sketch in these situations whether or not you are asked to.

Example 5

Find T, the point which divides the line PQ joining P(–1, 5, 0) and Q(9, 0, 20) in the ratio 3 : 2

Start with a sketch



From the sketch it is easy to see that $\frac{PT}{TQ} = \frac{3}{2}$ so that $2\,\overrightarrow{PT} = 3\,\overrightarrow{TQ}$

Substituting position vectors, $\quad 2\,(\mathbf{t} - \mathbf{p}) = 3\,(\mathbf{q} - \mathbf{t})$

Which after multiplying out and rearranging gives $\quad 5\mathbf{t} = 2\mathbf{p} + 3\mathbf{q}$ (check it out for yourself)

and from there it is simple to substitute the components of **p** and **q** and solve to get $\quad \mathbf{t} = \begin{pmatrix}5\\2\\12\end{pmatrix}$

so that T is the point (5, 2, 12)

You can use the formula $\mathbf{p} = \frac{n\mathbf{a} + m\mathbf{b}}{n + m}$ to find **p**, the position vector of P, the point which divides the line AB in the ratio m : n. Basically it is a rearrangement of the procedure above into a neat formula. If you find it easier you can use it with *x*, *y* and *z* components in turn instead of with the whole vector at once.

DON'T FORGET

It doesn't matter whether you use this formula or not – but you will have to know it if you are going to use it – it doesn't appear in the exam formula list.

LET'S THINK ABOUT THIS

If in example 5 above you were asked to find S where $\overrightarrow{PS} = 2\overrightarrow{PQ}$, you would be finding a point lying outside the line PQ. Try sketching it and working out the coordinates of S.